

Random algebraic geometry, attractors and flux vacua

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1 Introduction

A classic question in probability theory, studied by M. Kac, S. O. Rice and many others, is to find the expected number and distribution of zeroes or critical points of a random polynomial. The same questions can be asked for random holomorphic functions or sections of bundles, and are the subject of “random algebraic geometry.”

While this theory has many physical applications, in this article we focus on a variation on a standard question in the theory of disordered systems. This is to find the expected distribution of minima of a potential function randomly chosen from an ensemble, which might be chosen to model a crystal with impurities, a spin glass, or another disordered system. Now whereas standard potentials are real-valued functions, analogous functions in supersymmetric theories, such as the superpotential and the central charge, are holomorphic sections of a line bundle. Thus one is interested in finding the distribution of critical points of a randomly chosen holomorphic section.

Two related and much-studied problems of this type are the problem of finding attractor points in the sense of Ferrara, Kallosh and Strominger, and the problem of finding flux vacua as posed by Giddings, Kachru and Polchinski. These problems involve a good deal of fascinating mathematics and are good illustrations of the general theory.

2 Elementary random algebraic geometry

Let us introduce this subject with the problem of finding the expected distribution of zeroes of a random polynomial,

$$f(z) = c_0 + c_1 z + \dots c_N z^N.$$

We define a **random polynomial** to be a probability measure on a space of polynomials. A natural choice might be independent Gaussian measures on the coefficients,

$$d\mu[f] = d\mu[c_0, \dots, c_N] = \prod_{i=0}^N d^2 c_i \frac{\sigma_i}{2\pi} e^{-|c_i|^2/2\sigma_i^2}. \quad (1)$$

We still need to choose the variances. At first the most natural choice would seem to be equal variance for each coefficient, say $\sigma_i = 1/2$. We can characterize this ensemble by its two-point function,

$$\begin{aligned} G(z_1, \bar{z}_2) &\equiv \mathbb{E}[f(z_1) f^*(\bar{z}_2)] = \int d\mu[f] f(z_1) f^*(\bar{z}_2) \\ &= \sum_{n=0}^N (z_1 \bar{z}_2)^n \\ &= \frac{1 - z_1^{N+1} \bar{z}_2^{N+1}}{1 - z_1 \bar{z}_2} \end{aligned}$$

We now define $d\mu_0(z)$ to be a measure with unit weight at each solution of $f(z) = 0$, such that its integral over a region in \mathbb{C} counts the expected number of zeroes in that region. It can be written in terms of the standard Dirac delta function, by multiplication by a Jacobian factor,

$$d\mu_0(z) = \mathbb{E}[\delta^{(2)}(f(z)) \partial f(z) \bar{\partial} f^*(\bar{z})]. \quad (2)$$

To compute this expectation value, we introduce a constrained two-point function,

$$G_{f(z)=0}(z_1, \bar{z}_2) = \frac{\mathbb{E}[\delta^{(2)}(f(z)) f(z_1) f^*(\bar{z}_2)]}{\mathbb{E}[\delta^{(2)}(f(z))]}$$

It could be explicitly computed by using the constraint $f(z) = 0$ to solve for a coefficient c_i in the Gaussian integral, *i.e.* projecting on the linear subspace $0 = \sum c_i z^i$. The result, in terms of $G(z_1, \bar{z}_2)$, is

$$\mathbb{E}[\delta^{(2)}(f(z))] = \frac{1}{\pi G(z, \bar{z})};$$

$$G_{f(z)=0}(z_1, \bar{z}_2) = G(z_1, \bar{z}_2) - \frac{G(z_1, \bar{z})G(z, \bar{z}_2)}{G(z, \bar{z})}.$$

as can be verified by considering

$$\mathbb{E}[\delta^{(2)}(f(z)) f(z) f^*(\bar{z}_2)] \propto G_{f(z)=0}(z, \bar{z}_2) = G(z, \bar{z}_2) - \frac{G(z, \bar{z})G(z, \bar{z}_2)}{G(z, \bar{z})} = 0$$

Eq. (2) follows from this simply by taking derivatives:

$$\begin{aligned} d\mu_0(z) &= \frac{1}{G(z, \bar{z})} \lim_{z_1, z_2 \rightarrow z} D_1 \bar{D}_2 G_z(z_1, \bar{z}_2) \\ &= \frac{1}{\pi} \partial \bar{\partial} \log G(z, \bar{z}). \end{aligned}$$

For the constant variance ensemble Eq. (2),

$$d\mu_0(z) = \frac{d^2 z}{\pi} \left(\frac{1}{(1 - z\bar{z})^2} - \frac{(N+1)^2 (z\bar{z})^N}{(1 - (z\bar{z})^{N+1})^2} \right). \quad (3)$$

We see that as $N \rightarrow \infty$, the zeroes concentrate on the unit circle $|z| = 1$ (Hammersley, 1954).

A similar formula can be derived for the distribution of roots of a real polynomial on the real axis, using $d\mu(t) = \mathbb{E}[\delta(f(t))|df/dt|]$. One obtains (Kac, 1943):

$$d\mu_0^r(t) = \frac{dt}{\pi} \sqrt{\frac{1}{(1-t^2)^2} - \frac{(N+1)^2 t^{2N}}{(1-t^{2N+2})^2}}.$$

Integrating, one finds the expected number of real zeroes of a degree N random real polynomial is $E_N \sim \frac{2}{\pi} \log N$, and as $N \rightarrow \infty$ the zeroes are concentrated at $t = \pm 1$.

While concentration of measure is a fairly generic property for random polynomials, it is by no means universal. Let us consider another Gaussian ensemble, with variance $\sigma_n = N!/n!(N-n)!$. This choice leads to a particularly simple two-point function,

$$G(z, \bar{z}) = (1 + z\bar{z})^N, \quad (4)$$

and the distribution of zeroes

$$d\mu_0 = \frac{1}{\pi} \partial \bar{\partial} \log G = \frac{Nd^2 z}{\pi(1 + z\bar{z})^2}. \quad (5)$$

Rather than concentrate the zeroes, in this ensemble zeroes are uniformly distributed according to the volume of the Fubini-Study ($SU(2)$ -invariant) Kähler metric

$$\omega = \partial\bar{\partial}K; \quad K = \log(1 + z\bar{z})$$

on complex projective space \mathbb{CP}^1 .

We can better understand the different behaviors in our two examples by focusing on a hermitian inner product (f, g) on function space, associated to the measure Eq. (1) by the formal expression

$$d\mu[f] = [Df]e^{-(f,f)}.$$

In making this precise, let us generalize a bit further and allow f to be a holomorphic section of a line bundle \mathcal{L} , say $\mathcal{O}(N)$ over \mathbb{CP}^1 in our examples. We then choose an orthonormal basis of sections $(s_i, s_j) = \delta_{ij}$, and write

$$f \equiv \sum_i c_i s_i \tag{6}$$

and

$$d\mu[f] = \frac{1}{(2\pi)^N} \prod_{i=1}^N d^2 c_i e^{-|c_i|^2/2}.$$

We can then compute the two-point function

$$G(z_1, \bar{z}_2) \equiv \mathbb{E}[s(z_1)s^*(\bar{z}_2)] = \sum_{i=1}^N s_i(z_1)s_i^*(\bar{z}_2). \tag{7}$$

and proceed as before.

In these terms, the simplest way to describe the measure for our first example is that it follows from the inner product on the unit circle,

$$(f, g) = \oint_{|z|=1} \frac{dz}{2\pi z} f^*(z)g(z).$$

Thus we might suspect that this has something to do with the concentration of Eq. (3) on the unit circle. Indeed, this idea is made precise and generalized in (Shiffman and Zelditch, 2003).

Our second example belongs to a class of problems in which \mathcal{M} is compact and \mathcal{L} positive. In this case, the space $H^0(\mathcal{M}, \mathcal{L})$ of holomorphic sections is

finite dimensional, so we can take the basis to consist of all sections. Then, if \mathcal{M} is in addition Kähler, we can derive all the other data from a choice of hermitian metric $h(f, g)$ on \mathcal{L} . In particular, this determines a Kähler form ω as the curvature of the metric compatible connection, and thus a volume form $\text{Vol}_\omega = \omega^n/n!$. We then define the inner product to be

$$(f, g) = \int_{\mathcal{M}} \text{Vol}_\omega h(f, g).$$

Thus, the measure Eq. (1) and the final distribution Eq. (2) are entirely determined by h . In these terms, the underlying reason for the simplicity of Eq. (5) is that we started with the $SU(2)$ invariant metric h , so the final distribution must be invariant as well. More generally, Eq. (7) is a Szegő kernel. Taking $\mathcal{L} = \mathcal{L}_1^{\otimes N}$ for N large, this has a known asymptotic expansion, enabling a rather complete treatment (Zelditch, 2001).

Our two examples also make the larger point that a wide variety of distributions are possible. Thus we must put in some information about the ensemble of random polynomials or sections which appear in the problem at hand, to get convincing results.

The basic computation we just discussed can be vastly generalized: to multiple variables, multipoint correlation functions, many different ensembles, and different counting problems. We will discuss the distribution of critical points of holomorphic sections below.

3 The attractor problem

We now turn to our physical problems. Both are posed in the context of compactification of the type IIB superstring theory on a Calabi-Yau three-fold M . This leads to a four dimensional effective field theory with $N = 2$ supersymmetry, determined by the geometry of M .

Let us begin by stating the attractor problem mathematically, and afterwards give its physical background. We begin by reviewing a bit of the theory of Calabi-Yau manifolds. By Yau's proof of the Calabi conjecture, the moduli space of Ricci-flat metrics on M is determined by a choice of complex structure on M , denote this J , and a choice of Kähler class. Using deformation theory, it can be shown that the moduli space of complex structures, denote

this $\mathcal{M}_c(M)$, is locally a complex manifold of dimension $h^{2,1}(M)$. A point J in $\mathcal{M}_c(M)$ picks out a holomorphic three-form $\Omega_J \in H^{3,0}(M, \mathbb{C})$, unique up to an overall choice of normalization. The converse is also true; this can be made precise by defining the **period map** $\mathcal{M}_c(M) \rightarrow \mathbb{P}(H^3(M, \mathbb{Z}) \otimes \mathbb{C})$ to be the class of Ω in $H^3(M, \mathbb{Z}) \otimes \mathbb{C}$ up to projective equivalence. One can prove that the period map is injective (the Torelli theorem), locally in general and globally in certain cases such as the quintic in \mathbb{CP}^4 .

Now, the data for the attractor problem is a **charge**, a class $\gamma \in H^3(M, \mathbb{Z})$. An **attractor point** for γ is then a complex structure J on M such that

$$\gamma \in H_J^{3,0}(M, \mathbb{C}) \oplus H_J^{0,3}(M, \mathbb{C}). \quad (8)$$

This amounts to $h^{2,1}$ complex conditions on the $h^{2,1}$ complex structure moduli, so picks out isolated points in $\mathcal{M}_c(M)$, the attractor points.

There are many mathematical and physical questions one can ask about attractor points, and it would be very interesting to have a general method to find them. As emphasized by G. Moore, this is one of the simplest problems arising from string theory in which integrality (here due to charge quantization) plays a central role, and thus it provides a natural point of contact between string theory and number theory. For example, one might suspect that attractor Calabi-Yau's are arithmetic, *i.e.* are projective varieties whose defining equations live in an algebraic number field. This can be shown to always be true for $K3 \times T^2$, and there are conjectures about when this is true more generally (Moore, 2004).

A simpler problem is to characterize the distribution of attractor points in $\mathcal{M}_c(M)$. As these are infinite in number, one must introduce some control parameter. While the first idea which might come to mind is to bound the magnitude of γ , since the intersection form on $H^3(M, \mathbb{Z})$ is antisymmetric, there is no natural way to do this. A better way to get a finite set is to bound the period of γ , and consider the attractor points satisfying

$$Z_{max}^2 \geq |Z(\gamma; z)|^2 \equiv \frac{|\int_M \gamma \wedge \Omega|^2}{\int_M \Omega \wedge \bar{\Omega}}. \quad (9)$$

As an example of the type of result we will discuss below, one can show that for large Z_{max} , the density of such attractor points asymptotically approaches the Weil-Petersson volume form on \mathcal{M}_c .

We now briefly review the origins of this problem, in the physics of 1/2 BPS black holes in $N = 2$ supergravity. We begin by introducing local complex coordinates z^i on $\mathcal{M}_c(M)$. Physically, these can be thought of as massless complex scalar fields. These sit in vector multiplets of $N = 2$ supersymmetry, so there must be $h^{2,1}(M)$ vector potentials to serve as their bosonic partners under supersymmetry. These appear because the massless modes of the type IIB string include various higher rank p -form gauge potentials, in particular a self-dual four-form which we denote C . Self-duality means that $dC = *dC$ up to non-linear terms, where $*$ is the Hodge star operator in ten dimensions.

Now, Kaluza-Klein reduction of this four-form potential produces $b^3(M)$ one-form vector potentials A_I in four dimensions. Given an explicit basis of three-forms ω_I for $H^3(M, \mathbb{R}) \cap H^3(M, \mathbb{Z})$, this follows from the decomposition

$$C = \sum_{I=1}^{b_3} A_I \wedge \omega_I + \text{massive modes}.$$

However, because of the self-duality relation, only half of these vector potentials are independent; the other half are determined in terms of them by four-dimensional electric-magnetic duality. Explicitly, given the intersection form η_{ij} on $H^3 \otimes H^3$, we have

$$dA_i = \eta_{ij} *_4 dA_j \tag{10}$$

where $*_4$ denotes the Hodge star in $d = 4$. Thus we have $h^{2,1} + 1$ independent vector potentials. One of these sits in the $N = 2$ supergravity multiplet, and the rest are the correct number to pair with the complex structure moduli.

We now consider 1/2 BPS black hole solutions of this four dimensional $N = 2$ theory. Choosing any S^2 which surrounds the horizon, we can define the charge γ as the class in $H^3(M, \mathbb{Z})$ which reproduces the corresponding magnetic charges

$$Q_i = \frac{1}{2\pi} \int_{S^2} dA_i \equiv \int_M \omega_i \wedge \gamma.$$

Using Eq. (10), this includes all charges.

One can show that the mass M of any charged object in supergravity satisfies a **BPS bound**,

$$M^2 \geq |Z(\gamma; z)|^2. \tag{11}$$

The quantity $|Z(\gamma; z)|^2$, defined in Eq. (9), depends explicitly on γ , and implicitly on the complex structure moduli z through Ω . A 1/2 **BPS solution** by definition saturates this bound.

We now explain the “attractor paradox.” According to Bekenstein and Hawking, the entropy of any black hole is proportional to the area of its event horizon. This area can be found by finding the black hole as an explicit solution of four-dimensional supergravity, which clearly depends on the charge γ . In fact, we must fix boundary conditions for all the fields at infinity, in particular the complex structure moduli, to get a particular black hole solution. Now, normally varying the boundary conditions varies all the data of a solution in a continuous way. On the other hand, if the entropy has any microscopic interpretation as the logarithm of the number of quantum states of the black hole, one would expect e^S to be integrally quantized. Thus, it must remain fixed as the boundary conditions on complex structure moduli are varied, in contradiction with naive expectations for the area of the horizon, and seemingly contradicting Bekenstein and Hawking.

The resolution of this paradox is the **attractor mechanism**. Let us work in coordinates for which the four-dimensional metric takes the form

$$ds^2 = -f(r)dt^2 + dr^2 + \frac{A(r)}{4\pi}d\Omega_{S^2}^2.$$

With some work, one can see that in the 1/2 BPS case, the equations of motion imply that as r decreases, the complex structure moduli z follow gradient flow with respect to $|Z(\gamma, z)|^2$ in Eq. (11), and the area $A(r)$ of an S^2 at radius r decreases. Finally, at the horizon, z reaches a value z_* at which $|Z(\gamma, z_*)|^2$ is a local minimum, and the area of the event horizon is $A = 4\pi|Z(\gamma, z_*)|^2$. Since z_* is determined by minimization, this area will not change under small variations of the initial z , resolving the paradox.

A little algebra shows that the problem of finding non-zero critical points of $|Z(\gamma, z)|^2$, is equivalent to that of finding critical points $D_i Z = 0$ of the period associated to γ ,

$$Z = \int_M \gamma \wedge \Omega \tag{12}$$

usually called the **central charge**, with respect to the covariant derivative

$$D_i Z = \partial_i Z + (\partial_i K)Z. \tag{13}$$

Here

$$e^{-K} \equiv \int \Omega \wedge \bar{\Omega}. \quad (14)$$

The mathematical significance of this rephrasing is that K is a Kähler potential for the Weil-Peterson Kähler metric on $\mathcal{M}_c(M)$, with Kähler form $\omega = \partial\bar{\partial}K$, and Eq. (13) is the unique connection on $H^{(3,0)}(M, \mathbb{C})$ regarded as a line bundle over $\mathcal{M}_c(M)$, whose curvature is $-\omega$. These facts can be used to show that $D_i\Omega$ provides a basis for $H^{(2,1)}(M, \mathbb{C})$, so that the critical point condition forces the projection of γ on $H^{(2,1)}$ to vanish. This justifies our original definition Eq. (8).

4 Flux vacua in IIB string theory

We will not describe our second problem in as much detail, but just give the analogous final formulation. In this problem, a “choice of flux” is a pair of elements of $H^3(M, \mathbb{Z})$, or equivalently a single element

$$F \in H^3(M, \mathbb{Z} \oplus \tau\mathbb{Z}), \quad (15)$$

where $\tau \in \mathcal{H} \equiv \{\tau \in \mathbb{C} | \text{Im } \tau > 0\}$ is the so-called “dilaton-axion.”

A **flux vacuum** is then a choice of complex structure J and τ for which

$$F \in H_J^{3,0}(M, \mathbb{C}) \oplus H_J^{1,2}(M, \mathbb{C}). \quad (16)$$

Now we have $h^{2,1} + h^{0,3} = h^{2,1} + 1$ complex conditions on the joint choice of $h^{2,1}$ complex structure moduli and τ , so this condition also picks out special points, now in $\mathcal{M}_c \times \mathcal{H}$.

The critical point formulation of this problem is that of finding critical points of

$$W = \int \Omega \wedge F \quad (17)$$

under the covariant derivatives Eq. (13) and

$$D_\tau W = \partial_\tau W + (\partial_\tau W)Z$$

with K the sum of Eq. (14) and the Kähler potential $-\log \text{Im } \tau$ for the metric on the upper half plane of constant curvature -1 .

This is a sort of complexified version of the previous problem and arises naturally in IIB compactification by postulating a non-zero value F for a certain three-form gauge field strength, the **flux**. The quantity Eq. (17) is the superpotential of the resulting $N = 1$ supergravity theory, and it is a standard fact in this context that supersymmetric vacua (critical points of the effective potential) are critical points of W in the sense we just stated.

We can again pose the question of finding the distribution of flux vacua in $\mathcal{M}_c(M) \times \mathcal{H}$. Besides $|W|^2$, which physically is one of the contributions to the vacuum energy, we can also use the “length of the flux”

$$L = \frac{1}{\text{Im } \tau} \int \text{Re } F \wedge \text{Im } F \quad (18)$$

as a control parameter, and count flux vacua for which $L \leq L_{max}$. In fact, this parameter arises naturally in the actual IIB problem, as the “orientifold three-plane charge.”

What makes this problem particularly interesting physically is that it (and its analogs in other string theories) may bear on the solution of the cosmological constant problem. This begins with Einstein’s famous observation that the equations of general relativity admit a one parameter generalization,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} + \Lambda g_{\mu\nu}.$$

Physically, the cosmological constant Λ is the vacuum energy, which in our flux problem takes the form $\Lambda = \dots - 3|W|^2$ (the other terms are inessential for us here).

Cosmological observations tell us that Λ is very small, of the same order as the energy of matter in the present era, about $10^{-122}M_{Planck}^4$ in Planck units. However, in a generic theory of quantum gravity, including string theory, quantum effects are expected to produce a large vacuum energy, *a priori* of order M_{Planck}^4 . Finding an explanation for why the theory of our universe is in this sense non-generic is the cosmological constant problem.

One of the standard solutions of this problem is the “anthropic solution,” initiated in work of Weinberg and others, and discussed in string theory in (Bousso and Polchinski, 2000). Suppose that we are discussing a theory with a large number of vacuum states, all of which are otherwise candidates to describe our universe, but which differ in Λ . If the number of these vacuum

states were sufficiently large, the claim that a few of these states realize a small Λ would not be surprising. But one might still feel a need to explain why our universe is a vacuum with small Λ , and not one of the multitude with large Λ .

The anthropic argument is that, according to accepted models for early cosmology, if the value of $|\Lambda|$ were even 100 times larger than what is observed, galaxies and stars could not form. Thus, the known laws of physics guarantee that we will observe a universe with Λ within this bound; it is irrelevant whether other possible vacuum states “exist” in any sense.

While such anthropic arguments are controversial, one can avoid them in this case by simply asking whether or not any vacuum state fits the observed value of Λ . Given a precise definition of vacuum state, this is a question of mathematics. Still, answering it for any given vacuum state is extremely difficult, as it would require computing Λ to 10^{-122} precision. But it is not out of reach to argue that out of a large number of vacua, some of them are expected to realize small Λ . For example, if we could show that the number of otherwise physically acceptable vacua was larger than 10^{122} , and that the distribution of Λ among these was approximately uniform over the range $(-M_{Planck}^4, M_{Planck}^4)$, we would have made a good case for this expectation.

This style of reasoning can be vastly generalized, and given favorable assumptions about the number of vacua in a theory, could lead to falsifiable predictions independent of any *a priori* assumptions about the choice of vacuum state (Douglas, 2003).

5 Asymptotic counting formulae

We have just defined two classes of physically preferred points in the complex structure moduli space of Calabi-Yau threefolds, the attractor points and the flux vacua. Both have simple definitions in terms of Hodge structure, Eq. (8) and Eq. (16), and both are also critical points of integral periods of the holomorphic three-form.

This second phrasing of the problem suggests the following language. We define a **random period** of the holomorphic three-form to be the period for a randomly chosen cycle in $H_3(M, \mathbb{Z})$ of the types we just discussed (real or complex, and with the appropriate control parameters). We are then

interested in the expected distribution of critical points for a random period. This brings our problem into the framework of random algebraic geometry.

Before proceeding to use this framework, let us first point out some differences with the toy problems we discussed. First, while Eq. (12) and Eq. (17) are sums of the form Eq. (6), we take not an orthonormal basis but instead a basis s_i of integral periods of Ω . Second, the coefficients c_i are not normally distributed but instead drawn from a discrete uniform distribution, *i.e.* correspond to a choice of γ in $H^3(M, \mathbb{Z})$ or F as in Eq. (15), satisfying the bounds on $|Z|$ or L . Finally, we do not normalize the distribution (which is thus not a probability measure) but instead take each choice with unit weight.

These choices can of course be modified, but are made in order to answer the question, how many attractor points (or flux vacua) sit within a specified region of moduli space. The answer we will get is a density $\mu(Z_{max})$ or $\mu(L_{max})$ on moduli space, such that as the control parameter becomes large, the number of critical points within a region R asymptotes to

$$\mathcal{N}(R; Z_{max}) \sim \int_R \mu(Z_{max}).$$

The key observation is that to get such asymptotics, we can start with a Gaussian random element of $H^3(M, \mathbb{R})$ (or flux). In other words, we neglect the integral quantization of the charge or flux. Intuitively, this might be expected to make little difference in the limit that the charge or flux is large, and in fact one can prove that this simplification reproduces the leading large L or $|Z|$ asymptotics for the density of critical points, using standard ideas in lattice point counting.

This justifies starting with a two-point function like Eq. (7). While the integral periods s_i of Ω can be computed in principle (and have been in many examples) by solving a system of linear PDE's, the Picard-Fuchs equations, it turns out that one does not need such detailed results. Rather, one can use the following ansatz for the two-point function,

$$\begin{aligned} G(z_1, \bar{z}_2) &= \sum_{I=1}^{b_3} \eta^{IJ} s_I(z_1) s_J^*(\bar{z}_2) \\ &= \int_M \Omega(z_1) \wedge \bar{\Omega}(\bar{z}_2) \\ &= \exp -K(z_1, \bar{z}_2). \end{aligned}$$

In words, the two-point function is the formal continuation of the Kähler potential on $\mathcal{M}_c(M)$ to independent holomorphic and antiholomorphic variables. This incorporates the quadratic form appearing in Eq. (18) and can be used to count sections with such a bound.

We can now follow the same strategy as before, by introducing an expected density of critical points,

$$d\mu(z) = \mathbb{E}[\delta^{(n)}(D_i s(z)) \delta^{(n)}(\bar{D}_i \bar{s}(\bar{z})) \mid \det_{1 \leq i, j \leq 2n} H_{ij}], \quad (19)$$

where the “complex Hessian” H is the $2n \times 2n$ matrix of second derivatives

$$H \equiv \begin{pmatrix} \partial_i \bar{D}_{\bar{j}} \bar{s}(z) & \partial_i D_j s(z) \\ \bar{\partial}_{\bar{i}} \bar{D}_{\bar{j}} \bar{s}(z) & \bar{\partial}_{\bar{i}} D_j s(z) \end{pmatrix} \quad (20)$$

(note that $\partial Ds = DDs$ at a critical point). One can then compute this density along the same lines. The holomorphy of s implies that $\partial_i \bar{D}_{\bar{j}} s = \omega_{i\bar{j}} s$, which is one simplification. Other geometric simplifications follow from the fact that Eq. (19) depends only on s and a finite number of its derivatives at the point z .

For the attractor problem, using the identity

$$D_i D_j s = \mathcal{F}_{ijk} \omega^{k\bar{k}} \bar{D}_{\bar{k}} s = 0,$$

from special geometry of Calabi-Yau threefolds, the Hessian becomes trivial, and $\det H = |s|^{2n}$. One thus finds (Denef and Douglas, 2004) that the asymptotic density of attractor points with large $|Z| \leq Z_{max}$ in a region R is

$$\mathcal{N}(R, |Z| \leq Z_{max}) \sim \frac{2^{n+1}}{(n+1)\pi^n} Z_{max}^{n+1} \cdot \text{vol}(R)$$

where $\text{vol}(R) = \int_R \omega^n / n!$ is the volume of R in the Weil-Peterson metric. The total volume is known to be finite for Calabi-Yau threefold moduli spaces, and thus so is the number of attractor points under this bound.

The flux vacuum problem is complicated by the fact that DDs is non-zero and thus the determinant of the Hessian does not take a definite sign, and implementing the absolute value in Eq. (19) is nontrivial. The result (Douglas, Shiffman and Zelditch, 2004) is

$$\mu(z) \sim \frac{1}{b_3! \sqrt{\det \Lambda(z)}} \int_{\mathcal{H}(z) \times \mathbb{C}} |\det(HH^* - |x|^2 \cdot \mathbf{1})| e^{H^t \Lambda(z)^{-1} H - |x|^2} dH dx$$

where $\mathcal{H}(z)$ is the subspace of Hessian matrices Eq. (20) obtainable from periods at the point z , and $\Lambda(z)$ is a covariance matrix computable from the period data.

A simpler lower bound for the number of solutions can be obtained by instead computing the **index density**

$$\mu_I(z) = \mathbb{E}[\delta^{(n)}(D_i s) \delta^{(n)}(\bar{D}_i \bar{s}) \det_{1 \leq i, j \leq 2n} H_{ij}], \quad (21)$$

so-called because it weighs the vacua with a Morse-Witten sign factor. This admits a simple explicit formula (Ashok and Douglas, 2004),

$$I_{vac}(R, L \leq L_{max}) \sim \frac{(2\pi L_{max})^{b_3}}{\pi^{n+1} b_3!} \cdot \int_R \det(\mathcal{R} + \omega \cdot 1), \quad (22)$$

where \mathcal{R} is the $n+1 \times n+1$ dimensional matrix of curvature two-forms for the Weil-Petersson metric.

One might have guessed at this density by the following reasoning. If s had been a single-valued section on a compact \mathcal{M}_c (it is not), topological arguments determine the total index to be $[c_{n+1}(\mathcal{L} \otimes T^* \mathcal{M})]$, and this is the simplest density constructed solely from the metric and curvatures in the same cohomology class.

It is not in general known whether this integral over Calabi-Yau moduli space is finite, though this is true in examples studied so far. One can also control $|W|^2$ as well as other observables, and one finds that the distribution of $|W|^2$ among flux vacua is to a good approximation uniform. Considering explicit examples, the prefactor in Eq. (22) is of order 10^{100} – 10^{300} , so assuming that this factor dominates the integral, we have justified the Bousso-Polchinski solution to the cosmological constant problem in these models.

The finite L corrections to these formulae can be estimated using van der Corput techniques, and are suppressed by better than the naive $L^{-1/2}$ or $|Z|^{-1}$ one might have expected. However the asymptotic formulae for the numbers of flux vacuum break down in certain limits of moduli space, such as the large complex structure limit. This is because Eq. (18) is an indefinite quadratic form, and the fact that it bounds the number of solutions at all is somewhat subtle. These points are discussed at length in (Douglas, Shiffman and Zelditch, 2005).

Similar results have been obtained for a wide variety of flux vacuum counting problems, with constraints on the value of the effective potential at the minimum, on the masses of scalar fields, on scales of supersymmetry breaking, and so on. And in principle, this is just the tip of an iceberg, as the study of more or less any class of superstring vacua leads to similar questions of counting and distribution, less well understood at present. Some of these are discussed in (Douglas, 2003; Acharya et al 2005; Denef and Douglas 2005; Blumenhagen et al 2005).

Further reading

For background on random algebraic geometry and some of its other applications, as well as references in the text not listed here, consult Edelman and Kostlan, 1995 and Zelditch, 2001. The attractor problem is discussed in Ferrara et al 1995 and Moore, 2004, while IIB flux vacua were introduced in Giddings, Kachru and Polchinski 2002. Background on Calabi-Yau manifolds can be found in Cox and Katz 1999 and Gross et al 2003.

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